

Tropical constructive Pappus' theorem*

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Abstract

In this paper, we state a correspondence between classical and tropical Cramer's rule. This correspondence will allow us to compare linear geometric constructions in the projective and tropical spaces. In particular, we prove a constructive version of Pappus' theorem, as conjectured in [7].

1 Introduction

In the last years we have seen an increasing interest in tropical geometry. Introductory papers in tropical geometry may be found by the interested reader in Richter-Gebert et al. [7], the specific chapter in the book of Sturmfels [10] or the survey due to Mikhalkin [6]. In the last reference [6], G. Mikhalkin applies tropical geometry to enumerative geometry, proving a new way to calculate Gromov-Witten invariants in the projective plane. These invariants can be used to count the number of curves, with given genus and degree, passing through a configuration of points. This method was first suggested by Kontsevitch and it is also approached from another point of view in [8]. Also, in [4], we can find some computations of bounds for the Welschinger invariant in several toric surfaces using tropical geometry, which are interpreted as the algebraic count of real rational curves through a real configuration of points. Moreover, we can see an application of tropical geometry to combinatorics in [9]. Thus we observe that tropical geometry is a powerful tool to study different branches of mathematics. The problem is that it is not easy to translate familiar geometric definitions to a tropical framework.

This paper deals with the specific problem of successfully translating Pappus theorem to a suitable tropical state, using the notion of stable intersection and stable join, as presented in [7]. A more systematic study of tropical constructions and their relationship with classical ones is treated in [11].

We will work on the tropical semiring $(\mathbb{T}, \oplus, \odot) = (\mathbb{R}, \max, +)$, the set of real numbers with the tropical addition $a \oplus b = \max\{a, b\}$ and the tropical product $a \odot b = a + b$. In [7] and other references, it is also used the tropical semiring $(\mathbb{R}, \min, +)$ instead. But it is straightforward to check that all the results can be translated from one point of view to the other using the isomorphism $a \mapsto -a$.

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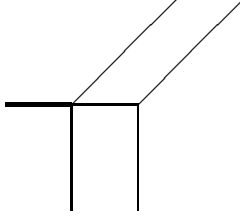


Figure 1: An infinite intersection of two lines

Now, we present the objects that we will work with. Given a tropical polynomial $f = \bigoplus_{i \in I} a_i \odot x^i$, where $i = (i_1, \dots, i_n)$ and $x^i = x_1^{i_1} \odot \dots \odot x_n^{i_n}$, we define the *tropical variety associated with f* as the set $\mathcal{T}(f) := \{x = (x_1, \dots, x_n) \in \mathbb{T}^n \mid f(x) = \max\{a_i + x_1 i_1 + \dots + x_n i_n, i \in I\} \text{ is attained for at least two different } i\}$. That is, the set of points where f is not differentiable.

In the following, we will use homogeneous coordinates in the tropical space \mathbb{T}^n , representing the point $(y_1, \dots, y_n) \in \mathbb{T}^n$ by $[y_1 : \dots : y_n : 0]$, with the identification $[y_1 : \dots : y_{n+1}] = [\alpha \odot y_1 : \dots : \alpha \odot y_{n+1}] = [\alpha + y_1 : \dots : \alpha + y_{n+1}]$, $\alpha \in \mathbb{T}$. We recover the affine coordinates using the usual subtraction (there is no notion of tropical subtraction), $[y_1 : \dots : y_n : y_{n+1}] = (y_1 - y_{n+1}, \dots, y_n - y_{n+1})$. We use homogeneous coordinates because it is easier to state Cramer's rule in this context.

In this direction, let us remark that the simplest well known varieties are tropical lines in the plane. For instance, take $f = a \odot x \oplus b \odot y \oplus c \odot z$ a linear homogeneous polynomial. The corresponding tropical line in \mathbb{T}^2 is the set $[x : y : z]$ such that $(a + x = b + y \geq c + z)$ or $(a + x = c + z \geq b + y)$ or $(b + y = c + z \geq a + x)$. We obtain three rays emerging from the point $[-a : -b : -c]$, with vectors in the directions $[0 : 0 : -1]$, $[0 : -1 : 0]$, $[-1 : 0 : 0]$.

Now we arrive to the following problem. What should be considered as the intersection of two given lines or, more generally, the intersection of tropical hypersurfaces. It is not trivial, as it may happen that two different lines share an infinite number of points, see figure (1). An answer is the following: given two tropical lines, there exists only one point in the intersection such that it is stable (in some sense) under small perturbations of the two lines, see [7]. This distinguished point is called the *stable intersection* of the lines. Similarly, given two points, there is only one line that passes through the two given points and is stable under small perturbation of the two points, it is called *the stable line* or the *stable join* of the points. Both stable intersection and stable join can be computed using the tropical analog of Cramer's rule, as follows.

First, the tropical determinant of a given $n \times n$ matrix in \mathbb{T} is defined as

$$\left| \begin{array}{ccc} a_{11} & \cdots & a_{1n} \\ \vdots & \cdots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{array} \right|_t = \bigoplus_{\sigma \in \Sigma_n} a_{1\sigma(1)} \odot \dots \odot a_{n\sigma(n)}.$$

Suppose now a tropical linear system of n equations in $n + 1$ homogeneous variables is given. We write O the $n \times (n + 1)$ matrix of coefficients and denote by O^i the matrix resulting from deleting the i -th column of O . Then, it is shown in [7] that the point $[|O^1|_t : \dots : |O^{n+1}|_t]$ is not only a common point of the n

On the other hand, [7] formulates a conjecture about the validity of the straightforward translation to the tropical context of a constructive version of Pappus' theorem. Here, of course, one must keep up, without modification in the tropical framework, with the given collection of construction steps. The goal of this paper is precisely to prove this conjecture. Let us introduce some notation needed to state it.

First of all, using duality, we identify the line $a \odot x \oplus b \odot y \oplus c \odot z$ with the point $[a : b : c]$ in \mathbb{T}^2 . The origin of the rays of the line defined by the polynomial is the point $[-a : -b : -c]$. Now, if we have two lines $[a : b : c]$, $[d : e : f]$, then the stable intersection corresponds to the point: $\left[\begin{bmatrix} b & c \\ e & f \end{bmatrix}_t : \begin{bmatrix} a & c \\ d & f \end{bmatrix}_t : \begin{bmatrix} a & b \\ d & e \end{bmatrix}_t \right]$, which is the stable solution of the system $a \odot x \oplus b \odot y \oplus c \odot z$, $d \odot x \oplus e \odot y \oplus f \odot z$. Also, if we have two points $[a : b : c]$, $[d : e : f]$, the previous expression corresponds to the coordinates of the stable line defined by those points. Thus, as in [7], we define the cross product of two points $x = [x_1 : x_2 : x_3]$ and $y = [y_1 : y_2 : y_3]$ as

$$x \otimes y = \left[\begin{bmatrix} x_2 & x_3 \\ y_2 & y_3 \end{bmatrix}_t : \begin{bmatrix} x_1 & x_3 \\ y_1 & y_3 \end{bmatrix}_t : \begin{bmatrix} x_1 & x_2 \\ y_1 & y_2 \end{bmatrix}_t \right],$$

which can be interpreted as the intersection of two lines or as finding the line through two points, depending on the context.

With this terminology theorem 3.2 states that there exists a tropical construction such that given five points 1, 2, 3, 4, 5 in the tropical plane, it computes three additional points 6, 7, 8 and nine lines $a, b, c, a', b', c', a'', b'', c''$ such that the hole set of elements is always in Pappus position (in the sense of [7]) and hence, the intersection of a'', b'', c'' is not empty.

Remark that in the thesis of the theorem, we do not mean that the three lines a'', b'', c'' share a point that is stable under perturbations but just that their intersection is non empty.

In our approach to proving this theorem it is essential to understand the behavior of Cramer's rule. In [7], Cramer's rule is analyzed using generic small perturbations in the coefficients of the system. Our point of view is a little bit different, as we try to compare the performance of Cramer's rule under the valuation map T . This allows us to give sufficient conditions for a chain of computations of tropical determinants to be lifted to the Puiseux series field. Using this lift, we are able to derive results from the classical to the tropical context.

The main idea is to take the input elements for the geometric construction in the tropical space, make a lift to the projective space over the Puiseux series field and perform there the given construction using classical Cramer's rule. Then we prove that, in this particular construction, if the elements in the lift are taken general enough, the results given by the application of Cramer's rule in the projective ambient should correspond with those obtained applying Cramer's rule in the tropical case.

Unfortunately, this procedure does not hold for other constructions, see counterexample 2.15. In fact, we can ensure that the good behavior in Pappus' case happens because our construction is of a very particular kind. Namely, we will prove that tropical and projective constructions behave well with respect to tropicalization when a certain graph associated with the construction is a tree.

The paper is structured as follows. In section 2 we will study with detail the relation between classical and tropical constructions, including a new proof (2.4)

of Cramer's rule. Then, we associate a graph to a tropical construction (2.8) and we introduce the notion of a *tropically admissible* construction of an elements (2.9). Finally we state (2.12) the validity of the specialization of a chain of Cramer's rule computations to the tropical space, when the construction graph is a tree. Section 3 is devoted to prove the conjectured version of Pappus' theorem (3.2), including some comments and remarks. We conclude (Section 4) with some reflections on the difficulty of achieving more general results on this topic.

2 Tropical Geometric Constructions

By a *geometric construction in the classical case* we will understand an abstract procedure consisting of

- Input data: A finite number of points or lines in $\mathbb{P}^2(\mathbb{K})$ that will eventually specialize to concrete elements given by its homogeneous coordinates (in the case of lines, the coordinates of the corresponding point in the dual plane).
- Allowed steps: computing the
 - line passing through two points
 - intersection point of two lines.
- Output: A finite set of points and lines

Likewise a *geometric construction in the tropical plane* consists of a similar procedure, replacing in the steps above the “line through two points” by the “stable line passing through two points” and the “intersection of two lines” by the “stable intersection of two lines”.

We want to study the relation between a given construction in the classical setting and the corresponding tropical one, see [11] for a more general study of tropical geometric constructions. Namely, we want to analyze, for different constructions, the commutativity of the following diagram:

$$\begin{array}{ccc}
 (\mathbb{K}^*)^2 & & \mathbb{T}^2 \\
 \text{Input} & \xleftarrow{T^{-1}} & \text{Input} \\
 \downarrow & & \downarrow \\
 \text{Output} & \xrightarrow{T} & \text{Output}
 \end{array} \tag{1}$$

where T stands for the tropicalization mapping. That is, given a construction, we want to study when, for some given tropical input data, we are able to find a suitable lift of the input data to the Puiseux series field \mathbb{K} , perform the construction in that projective plane, tropicalize all the output elements and find out that they are exactly the elements obtained by the tropical construction.

We will soon notice that it is not always possible. Even if it holds for some constructions, it will not do for every choice of an input lift (example 2.7). Let us start with the simplest case of one step constructions involving Cramer's rule only once.

For a Puiseux series $S = \alpha t^k + \dots$, $\alpha \neq 0$, we will denote by $Pc(S) = \alpha$ the *principal coefficient* of the series.

Let $B = (b_{i,j})$ be a $n \times n$ matrix in \mathbb{K}^* . Let us start by studying conditions for the commutativity of tropicalization and determinant computation, i.e. establishing when computing the determinant of B and tropicalizing it equals the determinant of $T(B)$. First, it can be easily checked ([11]) that this equality does not hold in general. Now, we will show that the conditions to have this property can be expressed in terms of the principal coefficients of the entries of B .

In this context we need to introduce the following terminology.

Definition 2.1. Let $O = (o_{ij})$ be a $n \times n$ matrix with coefficients in \mathbb{T} . Let $A = (a_{ij})$ be a $n \times n$ matrix in a ring R . We denote by $|O|_t$ the tropical determinant of O and we define

$$\Delta_O(A) = \sum_{\substack{\sigma \in \Sigma_n \\ o_{1,\sigma(1)} \odot \dots \odot o_{n,\sigma(n)} = |O|_t}} (-1)^{i(\sigma)} a_{1,\sigma(1)} \cdot \dots \cdot a_{n,\sigma(n)}$$

the *pseudo-determinant* of A with respect to O .

Lemma 2.2. Let $B = (b_{i,j})$ be a matrix in \mathbb{K}^* , $A = (a_{i,j})$ the matrix of principal coefficients in B and O the tropicalization matrix of B , $o_{i,j} = T(b_{i,j})$. If $\Delta_O(A) \neq 0$, then the principal coefficient of $|B|$, $Pc(|B|)$ equals $\Delta_O(A)$. Moreover $T(|B|)$ coincides with the tropical determinant $|O|_t$.

Proof. Notice that in the expansion of $|B|$, the permutations σ , where the order of the corresponding summand $b_{1,\sigma(1)} \cdot \dots \cdot b_{n,\sigma(n)}$ is the smallest possible one, are exactly the permutations in the expansion of $|O|_t$ where $|O|_t$ is attained. So, the coefficient of the term $t^{-|O|_t}$ is $\Delta_O(A)$. If it is non zero, then the order of $|B|$ is $-|O|_t$. \square

Now, we can extend this lemma to the context of Cramer's rule.

Definition 2.3. Let $O = (o_{ij})$ be a $n \times (n+1)$ tropical matrix. Let $A = (a_{ij})$ be a matrix in a ring R with the same dimension as O . We define

$$\text{Cram}_O(A) = (S_1, \dots, S_{n+1})$$

where $S_i = \Delta_{O^i}(A^i)$ and O^i (respectively, A^i) denote the corresponding submatrices obtained by deleting the i -th column in O (respectively, A).

Lemma 2.4. Suppose we are given a linear equation system in the semiring \mathbb{T} , with n equations in $n+1$ homogeneous variables. Let O be the coefficient matrix of the system. Let B be any matrix such that $T(B) = O$. Let A be the principal coefficient matrix of B . If no element of $\text{Cram}_O(A)$ vanishes, then the linear system defined by B has only one projective solution and its tropicalization equals the stable solution $[|O^1|_t : \dots : |O^{n+1}|_t]$

Proof. Apply the previous lemma to every component of the projective solution. \square

Proposition 2.5. If we have a one step construction, namely the stable join of two points or the stable intersection of two lines, then for every specialization of the input data, there exists a concrete lift that makes diagram (1) commutative.

Remark 2.6. Lemma 2.4 is not only useful to compute the intersection of n hyperplanes in \mathbb{T}^n , but it is also valid, for example, to compute the stable plane conic through five points, as it can be also interpreted as finding the stable intersection of 5 hyperplanes in the space of tropical plane conics \mathbb{T}^5 .

We apply the notation of 2.1 over the first coordinate data, yielding:

Therefore we cannot expect, in general, that the tropicalization of the generators of an ideal will describe the tropicalization of the variety this ideal generates.

Definition 2.8. Given a linear geometric construction, we associate a graph to every element P of the construction, that is, to all the points and lines appearing at some step of the construction, including input, intermediate and output elements. The vertices of the graph associated to an element P will correspond to all the elements that we have recursively used to construct P . We link every element with the elements from which it is constructed directly. That is, if point a (respectively line a) is the intersection of points b, c (respectively the join of points b and c), then we write edges ab and ac . We call this graph the *construction graph of P* .

Definition 2.9. We say that the construction of an element P is *tropically admissible (by Cramer’s rule)* if its associated construction graph is a tree.

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parametrizing the principal coefficients of the corresponding series with different variables. The following lemma describes the effect of the corresponding classical construction on this generic lift.

Lemma 2.10. *Let $C_i = \{c_i^1, \dots, c_i^{j_i}\}$, $1 \leq i \leq k$ be disjoint sets of variables. Suppose that we have $F_u = \{f_u^1, \dots, f_u^{n+1}\} \subseteq \mathbb{C}[\bigcup_{i=1}^k C_i]$, $1 \leq u \leq n$ sets of polynomials in the variables c_i^j . Suppose also that the following properties hold:*

- *For a fixed set F_u , f_u^l , with $1 \leq l \leq n+1$ are multihomogeneous polynomials in the sets of variables $C_{u^1}, \dots, C_{u^{n+1}}$ with the same multidegree.*
- *If $u \neq v$ then F_u, F_v involve different sets of variables C_i .*
- *In a family F_u , if $l \neq m$ then the monomials of f_u^l are all different from the monomials of f_u^m .*

Let us construct the $n \times (n+1)$ matrix

$$A = (f_u^l)_{\substack{1 \leq u \leq n, \\ 1 \leq l \leq n+1}}$$

and suppose that we are given a $n \times (n+1)$ matrix O in \mathbb{T} . Write

$$S = \text{Cram}_O(A) = (S_1, \dots, S_{n+1}).$$

Then

1. S_1, \dots, S_{n+1} are non-zero multihomogeneous polynomials in the sets of variables C_1, \dots, C_k with the same multidegree.
2. If σ, τ are different permutations in Σ_{n+1} which appear in the expansion of S_l (and, therefore $\sigma(n+1) = \tau(n+1) = l$), then all resulting monomials in $\prod_{u=1}^n (A^l)_u^{\sigma(u)}$ are different from the monomials in $\prod_{u=1}^n (A^l)_u^{\tau(u)}$
3. If $l \neq m$, then S_l, S_m have no common monomials.

Proof. First we prove 2. If we have two different permutations σ, τ , there is a natural number v , $1 \leq v \leq n$ where the permutations differ, then the monomials in $f_v^{\sigma(v)}, f_v^{\tau(v)}$ are all different and these polynomials are the only factors of the products $\prod_{u=1}^n (A^l)_u^{\sigma(u)}, \prod_{u=1}^n (A^l)_u^{\tau(u)}$ where we find the variables which appear in the family F_v . It follows that these products cannot share any monomial. In particular, in the sum of several of these products, there is no cancellation of monomials, proving item 1. So, in fact, we obtain that different minors share no monomial and we obtain immediately 3. All those minors must have the same multidegree, which is just the concatenation of the multidegree of the family F_1, \dots, F_n , by construction. \square

Example 2.11. At this point it may be helpful to give an example of the lemma. Consider the sets

$$C_1 = \{x, y\}, C_2 = \{z\}, C_3 = \{m, n\}, C_4 = \{o, p, q\}, C_5 = \{r\}.$$

$$F_1 = \{x^2yz + y^3z, x^3z, 2xy^2z\}$$

$$F_2 = \{mnor^2, m^2or^2 + mnpr^2, n^2or^2 + m^2pr^2 + n^2pr^2\}$$

Every polynomial in F_1 is multihomogeneous in C_1, C_2 with multidegree $(3, 1)$.

Every polynomial in F_2 is multihomogeneous in C_3, C_4, C_5 with multidegree

$(2, 1, 2)$.

All the monomials in the polynomial are different.

Then, matrix $A = \begin{pmatrix} x^2yz + y^3z & x^3z & 2xy^2z \\ mnor^2 & m^2or^2 + mnpr^2 & n^2or^2 + m^2pr^2 + n^2pr^2 \end{pmatrix}$.

We take as matrix O in $\text{Cram}_O(A)$, $O = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 3 & 2 \end{pmatrix}$

$$S_1 = (m^2or^2 + mnpr^2)(2xy^2z) = 2xy^2zm^2or^2 + 2xy^2zmnpr^2$$

$$S_2 = (x^2yz + y^3z)(n^2or^2 + m^2pr^2 + n^2pr^2) + (mnor^2)(2xy^2z) = x^2yzn^2or^2 + x^2yzm^2pr^2 + x^2yzn^2pr^2 + y^3zn^2or^2 + y^3zm^2pr^2 + y^3zn^2pr^2 + 2xy^2zmnor^2$$

$$S_3 = (x^2yz + y^3z)(m^2or^2 + mnpr^2) = x^2yzm^2or^2 + x^2yzmnpr^2 + y^3zm^2or^2 + y^3zmnpr^2.$$

Finally, we check that the polynomials S_1, S_2, S_3 share no monomial and are multihomogeneous in C_1, C_2, C_3, C_4, C_5 with multidegree $(3, 1, 2, 1, 2)$.

This lemma means that, as the polynomials S_i are never identically zero, there is always a suitable choice of the principal coefficients of the series involved in a lift such that the tropicalization of this lift agrees with the performed step of the tropical construction. Moreover, the output of the step (namely the polynomials S_i) can be considered as a single set F_u for a later construction. Clearly the input elements satisfy the restrictions of the lemma, as their principal coefficients are just different variables c_i^j . Thus, in the following theorem, we use induction in order to show that a tropically admissible construction agrees with the tropicalization of a projective construction. The following is the main theorem of the section.

Theorem 2.12 (General Lift). *Suppose we are given the geometric construction of elements q_1, \dots, q_s from elements p_1, \dots, p_n . Suppose that this construction can be meaningfully performed in the projective space by Cramer's rule. If the construction of each element q_i is tropically admissible by Cramer's rule then, for any specialization of the input tropical data given by homogeneous coordinates $p_i = [p_i^1 : \dots : p_i^{m_i}]$, $1 \leq i \leq n$, there exists a non empty set U in the space $(\mathbb{C}^*)^{m_1-1} \times \dots \times (\mathbb{C}^*)^{m_n-1}$ such that:*

1. *For every (x_1, \dots, x_n) in U there exist elements in the space of Puiseux series P_1, \dots, P_n such that $T(P_i) = p_i$, $Pc(P_i) = x_i$ and the projective construction of Q_1, \dots, Q_s from P_1, \dots, P_n is meaningful.*
2. *For all elements P_1, \dots, P_n in the multiprojective space $\mathbb{P}^{m_1-1}(\mathbb{K}) \times \dots \times \mathbb{P}^{m_n-1}(\mathbb{K})$ such that $T(P_i) = p_i$ and $(Pc(P_1), \dots, Pc(P_n)) \in U$, the tropicalization of the final elements of the construction Q_1, \dots, Q_s agree with the tropical elements q_1, \dots, q_s constructed using tropical determinants. That is, all lifts with principal coefficients in U yield to the same tropical final elements.*

Proof. We take generic projective lifts P_1, \dots, P_n with $T(P_i) = p_i$, writing $Pc(P_i) = [c_i^1 : \dots : c_i^{m_i}]$, indeterminate variables. Since each step of the construction is given by Cramer's rule and all the variables are different, we are in the hypotheses of 2.10, taking for the first step $f_i^j = c_i^j$, $F_i = \{f_i^1, \dots, f_i^{m_i}\}$, $C_i = \{c_i^1, \dots, c_i^{m_i}\}$. Each element of the construction is admissible by Cramer's rule. Any intermediate or output element is constructed from different objects. As this element is tropically admissible, the condition of its construction graph

being a tree corresponds to the fact that the input elements and hence the variables its parents depend are different, so we are still in the conditions of 2.10. This allows us to use induction in 2.10 because we will always have disjoint sets of variables on the rows of our matrices. So, all principal coefficients of all the steps in the construction will be non-zero multihomogeneous polynomials in the sets C_i .

We define U as the subset of $(\mathbb{C}^*)^{m_1-1} \times \dots \times (\mathbb{C}^*)^{m_n-1}$ where all these multihomogeneous polynomials do not vanish (considering $(\mathbb{C}^*)^{m_i-1} \subseteq \mathbb{P}^{m_i-1}(\mathbb{C})$ and taking homogeneous coordinates). If the principal coefficients of the P_i are in U , then we obtain along the construction that all the principal coefficients of the intermediate elements are non-zero. Then, by lemma 2.4, the tropicalization of each step will be exactly the corresponding tropical determinant, which is independent of the chosen lift P_i .

Of course, for x_1, \dots, x_n in U , one possible lift is $P_i = [x_i^1 t^{-p_i^1} : \dots : x_i^{m_i} t^{-p_i^{m_i}}]$. \square

Definition 2.13. Given a tropical geometric construction and a specialization of the input data, we call *general lift* of the input data any lift whose principal coefficients belong to the set U defined above.

Remark 2.14. Theorem 2.12 asserts that, for every tropical geometric construction and for every input data, if the construction graph of every element is a tree, then there exists a lift that agrees with our tropical construction, no matter what the input data is. Also, it is remarkable that this theorem is stated in general dimension, not just in the plane. So the result is valid for constructions in \mathbb{T}^n , the only restriction we have to consider is that of constructions involving only the stable intersection of n hyperplanes and the stable join of n points.

The following example shows what may happen if the construction graph is not a tree and we are not in the situation of theorem 2.12

Example 2.15. Suppose we are given a, b, c three points in the plane. Let $l_1 = \overline{ab}$, $l_2 = \overline{ac}$ be the lines through these points and $p = l_1 \cap l_2$. The construction of l_1 and l_2 is tropically admissible by Cramer's rule, but not the construction of p , because we have the cycle p, l_1, a, l_2, p . The problem is that we have used twice the point a in order to construct p . Firstly it is used in the construction of l_1 and then in the construction of l_2 .

So, after specialization, we may have some algebraic relations making a pseudo-determinant identically zero for every lift. For example, we take $a = [0 : 0 : 0]$, $b = [-2 : 1 : 0]$, $c = [-1 : 3 : 0]$. Tropically, the construction yields $l_1 = [1 : 0 : 1] = 1 \odot x \oplus 0 \odot y \oplus 1 \odot z$, $l_2 = [3 : 0 : 3] = 3 \odot x \oplus 0 \odot y \oplus 3 \odot z$ and finally $p = [3 : 4 : 3] = [0 : 1 : 0] \neq a$. But, for every lift of a, b, c such that the construction is well defined, the final element must be the lift of a . These lifts take the form $\tilde{a} = [a_1 : a_2 : a_3]$, $\tilde{b} = [b_1 t^2 : b_2 t^{-1} : b_3]$, $\tilde{c} = [c_1 t : c_2 t^{-3} : c_3]$, where terms of bigger degree in the series do not affect the result. In this case, $\tilde{l}_1 = [-a_3 b_2 t^{-1} + a_2 b_3 : -a_1 b_3 + a_3 b_1 t^2 : a_1 b_2 t^{-1} - a_2 b_1 t^2]$ and $\tilde{l}_2 = [-a_3 c_2 t^{-3} + a_2 c_3 : -a_1 c_3 + a_3 c_1 t : a_1 c_2 t^{-3} + a_2 c_1 t]$ which tropicalize correctly to l_1 and l_2 (as expected, because the construction graphs of l_1 and l_2 are trees). Now, we want to construct \tilde{p} . Here, $O = \begin{pmatrix} 1 & 0 & 1 \\ 2 & 0 & 2 \end{pmatrix}$ and $A = \begin{pmatrix} -a_3 b_2 & -a_1 b_3 & a_1 b_2 \\ -a_3 c_2 & -a_1 c_3 & a_1 c_2 \end{pmatrix}$. Now it is easy to see that $\Delta_{O^2} A^2 = -a_1 a_3 b_2 c_2 + a_1 a_3 b_2 c_2 = 0$. In fact, \tilde{p} must be \tilde{a} .

exists no condition on these elements in order to develop our construction and achieve the results.

4 Conclusions

In this paper we have explored the possibility of developing a tropical counterpart of classical geometric constructions. In view of theorem 2.12 we have succeeded for Pappus' theorem. But we have also shown through examples that there are several restrictions on the constructions to apply this theorem. It would be interesting to have some other remarkable examples of correct tropicalization of classical theorems.

On the other hand, proposition 2.5 shows that, via Cramer's rule, the situation is specially simple for one step constructions. We observe that this behavior is also present in some other successful applications of tropical geometry such as that of computing genus zero curves through a general configuration of points developed in [6]. This is, too, a "one step mathematics", since given a set of tropically general points, we "merely" construct the zero genus curves of given degree that passes through these points. The key problem seems to be handling tropical varieties constructed from other previously constructed varieties.

So the morale suggested by the results presented in this paper is that using tropical geometry is affordable (at this moment) when dealing with "one step mathematics", but it is not yet when dealing with geometric objects defined from other objects that are not free in some algebraic sense.

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